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On the sources of simple modules in nilpotent blocks

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Abstract

Let G be a finite group and let k be an algebraically closed field of characteristic p. If b is a nilpotent block of kG with defect group P, then there is a unique isomorphism class of simple kGb-modules and Puig proved that the source of this module is an endo-permutation kP-module. It is conjectured that the image of this source is always torsion in the Dade group.

Let H be a finite group and let P be a p-subgroup of $\mathrm{Aut}(H)$. Also let c be a defect zero block of kH. If c is P-stable and $\mathrm{Br}_P(c) \neq 0$, then c is a nilpotent block of $k(H \rtimes P)$ and $k(H \rtimes P)c$ has P as a defect group. In this paper, we will investigate the sources of the simple $k(H \rtimes P)c$-modules when $P \cong C_p \times C_p$. Suppose that we can find an H and c as above such that a source of a simple $k(H \rtimes P)c$-module is not torsion in the Dade group. Then we can find H and c as above with H a central p'-extension of a simple group. When $p \geqslant 3$ we show that H can be found in a quite restrictive subset of simple groups.

Keywords: Representation of finite groups; Block theory; Nilpotent blocks; Source algebras

1. Introduction

Puig showed that the source algebra of a nilpotent block is isomorphic to $\mathrm{End}_k(L) \otimes kP$ where P is a defect group of b and L is an indecomposable endo-permutation module with vertex P. We are interested in investigating which endo-permutation modules L can show up in this way.

Before we state the main result, we fix some notation. For this entire paper, p will be a fixed prime and k will be an algebraically closed field of characteristic p. Let G be a finite group.

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A *block* of kG is a primitive central idempotent of kG. If b is a block of kG, the algebra kGb is a *block algebra* of kG.

Let V be an indecomposable kG-module. A *vertex* of V is a minimal subgroup Q such that V is a direct summand of $\mathrm{Ind}_Q^G(W)$ for some kQ-module W. Since V is indecomposable, W can be taken to be indecomposable and is then called a *source* of V.

Associated with any block b of kG is a conjugacy class of p-groups called the defect groups of b. If 1 is a defect group of b we say that b is a *defect zero* block. Let b be a defect zero block of kG and let P be a p-subgroup of $\mathrm{Aut}(G)$. If b is stable under the action of P, then b remains a central idempotent of $k(G \rtimes P)$. It is easy to check that b is a block of $k(G \rtimes P)$. If we also assume that $\mathrm{Br}_P(b) \neq 0$, then P will be a defect group of b as a block of $k(G \rtimes P)$. The main result of this paper is the following.

Theorem 1.1. *Suppose that G is a finite group such that $P \cong C_p \times C_p \leqslant \mathrm{Aut}(G)$, and that b is a P-stable defect* 0 *block of kG such that $\mathrm{Br}_P(b) \neq 0$. Suppose also that the source V of a simple $k(G \rtimes P)b$-module M is a finitely generated non-torsion endo-permutation kP-module. Then we can find G, b, V with the above properties where G is a central p'-extension of a simple group.*

If we restrict to the case of odd p we have the following corollary.

Corollary 1.2. *Assume the setup of Theorem* 1.1. *If p is odd, then we can find G, b, V where G is one of the following*:

(i) (a) *G is a central p'-extension of $A_n(q)$ with $p \mid (n+1, q-1, f)$ where $q = r^f$; or*
    (b) *G is a central p'-extension of ${}^2A_n(q)$ with $p \mid (n+1, q+1, f)$ where $q = r^f$; or*
(ii) *$p = 3$, $q = r^f$ and G is a central extension of $D_4(q)$ with $3 \mid f$.*

This result is a reduction, for the above blocks, of the following conjectures of Puig and Feit to a central p'-extension of a simple group. A weaker version of Feit's conjecture can be found in [5].

Conjecture 1.3 *(Puig). Let P be a finite p-group. There are only finitely many isomorphism classes of interior P-algebras which are the source algebra of a block with P as a defect group.*

Conjecture 1.4 *(Feit). Let P be a finite p-group. There are only finitely many isomorphism classes of kP-modules which are sources of a simple module.*

Puig has proved both of these conjectures for p-solvable groups [14]. The dimensions of sources of simple modules of p-solvable groups with $C_p \times C_p$ as a p-Sylow subgroup are calculated in Muncke [10].

This paper is organized as follows. Section 2 recalls some basic definitions and theorems. Sections 3 and 4 contain some general reduction theorems. Section 5 contains the main reduction to central p'-extensions of simple groups establishing Theorem 1.1. In Section 6 we eliminate most of the simple groups. Finally, in Section 7, we eliminate $E_6(q)$ and ${}^2E_6(q)$ and obtain Corollary 1.2.

2. Preliminaries

Let G be a finite group. An *interior G-algebra* is a pair (A, φ) where A is a k-algebra and $\varphi : G \longrightarrow A^\times$ is a group homomorphism. φ is called the *structural homomorphism* of A and when φ is understood we will simply write A for the interior G-algebra (A, φ). We also use ${}^g a$ to denote $\varphi(g)a(\varphi(g))^{-1}$. If H is a subgroup of G and A is an interior G-algebra, the set of H fixed points of A is $\{a \in A \mid {}^h a = a \text{ for all } h \in H\}$ and is denoted A^H. The main examples of interior G-algebras are the group algebra kG and the block algebras kGb. These are interior G-algebras with structural homomorphisms sending g to its image, still denoted g, in kG and g to gb respectively.

Suppose that H is a subgroup of a finite group G and A is an interior G-algebra. The *relative trace* Tr_H^G is a map from A^H to A^G defined by $a \longmapsto \sum_{g \in [G/H]} {}^g a$. Also note that $\mathrm{Im}(\mathrm{Tr}_H^G)$ is an ideal of A^G which we denote as A_H^G. If P is a p-subgroup of G, the *Brauer quotient* $A(P)$ is defined to be $A^P / \sum_{Q \subsetneqq P} A_Q^P$ and the *Brauer homomorphism* $\mathrm{Br}_P^A : A^P \longrightarrow A(P)$ is defined to be the obvious map. If A is understood we will simply write Br_P for the Brauer homomorphism.

Let G be a finite group and let b be a block of kG. A *defect group* of b is a minimal subgroup P such that $b \in \mathrm{Im}(\mathrm{Tr}_P^G)$. This is equivalent to a maximal p-subgroup P such that $\mathrm{Br}_P(b) \neq 0$.

Now let b be a block of kG for a finite group G with defect group P. A primitive idempotent i of $(kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$ is a *source idempotent* of b. The algebra $ikGi$ is an interior P-algebra with structural homomorphism $u \longmapsto iui$ and is called a *source algebra* of b. Blocks are Morita equivalent to their source algebras and most of the local structure of the block can be recovered from a source algebra.

We now quickly recall some results on endo-permutation modules. Let P be a finite p-group. A kP-module M is an *endo-permutation* kP-module if $\mathrm{End}_k(M)$ is a permutation kP-module. Dade showed that for a given endo-permutation module M there is at most one isomorphism class of summands with vertex P [4]. If such a summand exists, we say M is *capped* and we call that summand a *cap* of M. Since the cap is unique up to isomorphism, we can define a relation on the set of capped endo-permutation kP-modules by $M \sim N$ if and only if M and N have isomorphic caps. It is easy to see that this is an equivalence relation and we denote the equivalence class of a capped endo-permutation kP-module M by $[M]$.

Dade also showed that if M and N are capped endo-permutation kP-modules then so is $M \otimes N$. So we can place an abelian group structure on the set of equivalence classes by defining $[M] + [N]$ to be $[M \otimes N]$. This group is the *Dade group* of P and is denoted $D(P)$. Puig showed that $D(P)$ is finitely generated [12]. This implies that the torsion part of $D(P)$ is finite.

Next we recall some definitions and results on Brauer pairs. Let G be a finite group. A *Brauer pair* is a pair (P, e) where P is a p-subgroup of G and e is a block of $kC_G(P)$. If (P, e) and (Q, f) are Brauer pairs, then we say (Q, f) is *contained* in (P, e) and write $(Q, f) \subseteq (P, e)$ if $Q \subseteq P$ and for any primitive idempotent $i \in (kG)^P$, if $\mathrm{Br}_P(i)e \neq 0$, then $\mathrm{Br}_Q(i)f \neq 0$. This provides a partial ordering on the set of Brauer pairs. Let b be a block of kG. A Brauer pair (P, e) is called a *b-Brauer pair* if and only if $\mathrm{Br}_P(b)e \neq 0$.

For a Brauer pair (P, e), the *normalizer of* (P, e) *in* G is defined to be $N_G(P, e) = \{g \in N_G(P) \mid {}^g e = e\}$. A block b is said to be *nilpotent* if $N_G(Q, e)/C_G(Q)$ is a p-group for every b-Brauer pair (Q, e). The main theorem on nilpotent blocks that we make use of is the following.

Theorem 2.1 *(Puig, [13]). Let G be finite group, let P be a finite p-group and let b be a nilpotent block of kG with P a defect group. If $i \in (kGb)^P$ is a source idempotent, then the source algebra*

$ikGi$ is isomorphic to $S \otimes_k kP$ as interior P algebras where S is a k-simple algebra isomorphic to $\mathrm{End}_k(L)$ *and L is an indecomposable endo-permutation kP-module with vertex P.*

Note that this implies there is a unique isomorphism class of simple kGb-modules and the sources of these modules will be isomorphic to L.

3. Reduction theorems

In this section, we state some of the main reduction theorems we will need. The following is well known and will allow us to make a significant reduction. A more general theorem that contains the following is 16.6 of [15].

Proposition 3.1. *Let G be a finite group, let N be a normal subgroup of G, let c be a block of kN, let b be a block of kG such that $bc \neq 0$ and let H be the stabilizer of c in G. Then*

- (i) *The map $\mathrm{Tr}_H^G : Z(kHc) \longrightarrow Z(kG\,\mathrm{Tr}_H^G(c))$ is an algebra isomorphism whose inverse is given by multiplication by c. In particular, we have a unique block d of kH such that $\mathrm{Tr}_H^G(d) = b$, $dc = d$ and $cbc = d$.*
- (ii) *kHc is Morita equivalent to $kG(\mathrm{Tr}_H^G(c))$.*
- (iii) *Every defect group of d is a defect group of b.*
- (iv) (*Puig*) *Every source algebra of kHd is a source algebra of kGb.*
- (v) *If b is nilpotent then d is nilpotent.*

Assume the setup of the previous proposition with b, and therefore d, nilpotent. If V is the unique (up to isomorphism) simple kHd-module, then $kGc \otimes_{kHc} V \cong kG \otimes_{kH} V = \mathrm{Ind}_H^G(V)$ is the unique (up to isomorphism) simple kGb-module. So if W is a source of V, then W will be a source of $\mathrm{Ind}_H^G(V)$.

Now let S be a finite dimensional matrix algebra over k on which a finite group G acts as algebra automorphisms. Details of the following can be found in Section 3 of [7]. The Skolem–Noether theorem tells us that for each $x \in G$ we can choose an $s_x \in S^\times$ such that ${}^x t = s_x t s_x^{-1}$ for all $t \in S$. Now we fix a choice of s_x for each $x \in G$. If $x, y \in G$, then $s_x s_y$ and s_{xy} will have the same action on S. So there must be a $\lambda(x, y) \in k^\times$ such that $s_x s_y = \lambda(x, y)s_{xy}$. Defining $\lambda(x, y)$ as above for each $x, y \in G$, the map $\lambda : G \times G \longrightarrow k^\times$ is a 2-cocycle of G with values in $k^\times$. If λ is any 2-cocycle of G with values in $k^\times$, the *twisted group algebra* $k_\lambda G$ is defined to be the free k-module with basis $\{\hat{x}\}_{x\in G}$ and multiplication being the bilinear extension of $\hat{x}\hat{y} = \lambda(x, y)\widehat{xy}$ for $x, y \in G$.

Let N be a normal subgroup of a finite group G and let c be a G-stable defect zero block of kN. Since kNc is simple we can define a 2-cocycle λ of G as above. We will choose the s_x so that λ induces a 2-cocycle of G/N. First we choose $\{1_G = x_1, \dots, x_t\} = [G/N]$ to be a set of coset representatives of G/N and for each of the x_i we choose an s_{x_i} as above. For $i = 1, \dots, t$ and $n \in N$, define $s_{x_i n}$ to be $s_{x_i} nc$. From this one can easily check that $\lambda(g, n) = 1$ and $\lambda(n, g) = 1$ for all $g \in G$ and $n \in N$. This implies that $\lambda(gn, hm) = \lambda(g, h)$ for all $g, h \in G$ and $n, m \in N$. Therefore we can define a 2-cocycle $\tilde{\lambda}(gN, hN) = \lambda(g, h)$. We will use λ to denote $\tilde{\lambda}$. The next result is essentially Puig's algebra theoretic version of Fong reduction of characters.

Theorem 3.2. *Let G be a finite group, let N be a normal subgroup of G and let c be a defect zero block of kN which is G-stable. Let $S = kNc$ and define the 2-cocycle λ of G and G/N as above. Then there is an algebra isomorphism*

$$\varphi : kGc \longrightarrow S \otimes_k k_{\lambda^{-1}}(G/N)$$

mapping xc to $s_x \otimes \widehat{xN}$ with inverse map ψ mapping $s \otimes \widehat{xN}$ to $ss_x^{-1}x$. Where $x \in G$, $s \in S$ and s_x are defined as above for each $x \in G$.

Proof. This is a routine verification that the above maps are homomorphisms and inverses of each other. $\square$

Since we are interested in source algebras, we want the isomorphisms in Theorem 3.2 to be an isomorphism of interior P-algebras. With the setup above, suppose that the p-group P is a subgroup of G such that $P \cap N = \{1\}$. Since P is a p-group, we know its action on $S = kNc$ is induced from a group homomorphism $\varphi : P \longrightarrow S^\times$ by 21.4 of [15]. We can then define $\psi : PN \longrightarrow S^\times$ by $\psi(rn) = \varphi(r)nc$ for $r \in P$ and $n \in N$. Since $\varphi(r)n(\varphi(r))^{-1} = rnr^{-1}$ for all $r \in P$ and $n \in N$, one can easily check that ψ is a group homomorphism.

Now pick $\{1 = x_1, \ldots, x_t\} = [G/PN]$ to be a set of coset representatives. Choose an s_{x_i} for each x_i as we did above and define $s_{x_i rn}$ to be $s_{x_i}\psi(rn) = s_{x_i}\varphi(r)nc$ for all $r \in P$ and $n \in N$. We will again get a 2-cocycle of G with values in $k^\times$ which induces a 2-cocycle of G/N with values in $k^\times$. With this new choice of s_g the map $r \longmapsto s_r$ is a homomorphism from P to $S^\times$. Also the map $r \longmapsto \widehat{rN}$ from P to $(k_{\lambda^{-1}}(G/N))^\times$ is a group homomorphism. By defining the structural homomorphism $r \longmapsto s_r \otimes_k \widehat{rN}$ for $r \in P$, we obtain an interior P-algebra structure on $S \otimes_k k_{\lambda^{-1}}(G/N)$. The following proposition is then immediate.

Proposition 3.3. *Let G be a finite group, let N be a normal subgroup of G, let c be a defect zero block of N which is G-stable and let $S = kNc$. Suppose that the p-group P is a subgroup of G such that $P \cap N = \{1\}$ and λ is a 2-cocycle defined as above. Then the k-algebra isomorphism $\varphi : kGc \longrightarrow S \otimes_k k_{\lambda^{-1}}(G/N)$ defined in Theorem 3.2 is an isomorphism of interior P-algebras.*

To eliminate the twisted algebra from the above theorems we will use the following theorem which can be found in 10.5 of [15]. We give a slightly more explicit proof.

Theorem 3.4. *If λ is a 2-cocycle of a finite group G with values in $k^\times$, then there exists a central extension of finite groups $1 \longrightarrow Z \longrightarrow H \longrightarrow G \longrightarrow 1$ where Z is cyclic of order prime to p such that $k_\lambda G \cong kHe$ for some idempotent e of kZ. If G is perfect, then H can be chosen to be perfect.*

The proof uses the following well known lemma whose proof can be found in [8].

Lemma 3.5. *With the setup of the theorem, there is a 2-cocycle α with values in the $|G|_{p'}$th roots of unity of $k^\times$ and a map $\beta : G \longrightarrow k^\times$ such that $\lambda(x, y) = \alpha(x, y)\beta(x)^{-1}\beta(y)^{-1}\beta(xy)$.*

Proof of Theorem 3.4. Let Z be the group of $|G|_{p'}$th roots of unity of $k^\times$. The lemma tells us there exists a 2-cocycle α and a map $\beta : G \longrightarrow k^\times$ such that $\lambda(x, y) = \alpha(x, y)\beta(x)\beta(y)\beta(xy)^{-1}$ with the values of α in Z. We define a central extension of G by Z by letting $H = G \times Z$ as a set and defining $(x,\eta)(y,\zeta) = (xy, \eta\zeta\alpha(x,y))$. This is a group since α is a 2-cocycle. Let $e = \frac{1}{|Z|}\sum_{\zeta\in Z}\zeta^{-1}(1,\zeta)$. Then

$$e^2 = \frac{1}{|Z|^2}\sum_{\zeta,\zeta'\in Z}(\zeta\zeta')^{-1}(1,\zeta\zeta') = e.$$

So e is an idempotent and it is clearly central. We now show that $k_\lambda G \cong kHe$.

Define $\varphi : kH \longrightarrow k_\lambda G$ to be the linear extension of $\varphi((x,\eta)) = \beta(x)\eta\hat{x}$. Then

$$\begin{aligned}\varphi\big((x,\eta)\big)\varphi\big((y,\xi)\big) &= \beta(x)\beta(y)\eta\xi\hat{x}\hat{y}\\ &= \beta(x)\beta(y)\eta\xi\lambda(x,y)\widehat{xy}\end{aligned}$$

and $\varphi((xy,\eta\xi\alpha(x,y))) = \beta(xy)\eta\xi\alpha(x,y)\widehat{xy}$ so φ is an algebra homomorphism and it is clearly onto. It is also easy to see $\varphi(e) = 1$ and so restricting φ to kHe is still onto. Now define $\psi : k_\lambda G \longrightarrow kHe$ by $\psi(\hat{g}) = \beta(g)^{-1}(g,1)e$. Note that if $\zeta' \in Z$ and $g \in G$, then

$$\zeta'(g,1)e = \frac{1}{|Z|}\sum_{\zeta\in Z}\zeta'\zeta^{-1}(g,\zeta) = \frac{1}{|Z|}\sum_{\zeta\in Z}\zeta^{-1}(g,\zeta'\zeta) = \frac{1}{|Z|}\sum_{\zeta\in Z}\zeta^{-1}(g,\zeta')(1,\zeta) = (g,\zeta')e.$$

From this we can verify that φ and ψ are inverses.

Then $\psi(\hat{x}\hat{y}) = \psi(\lambda(x,y)\widehat{xy}) = \lambda(x,y)\beta(xy)^{-1}(xy,1)e$ and

$$\begin{aligned}\psi(\hat{x})\psi(\hat{y}) &= \beta(x)^{-1}(x,1)\beta(y)^{-1}(y,1)e\\ &= \beta(x)^{-1}\beta(y)^{-1}\big(xy,\alpha(x,y)\big)e\\ &= \beta(x)^{-1}\beta(y)^{-1}\alpha(x,y)(xy,1)e.\end{aligned}$$

So ψ is a homomorphism. Also notice that

$$\begin{aligned}\varphi\big(\psi(\hat{x})\big) &= \varphi\big(\beta(x)^{-1}(x,1)e\big)\\ &= \beta(x)^{-1}\beta(x)\hat{x}\\ &= \hat{x}\end{aligned}$$

and

$$\begin{aligned}\psi\big(\varphi\big((x,\eta)e\big)\big) &= \psi\big(\beta(x)\eta\hat{x}\big)\\ &= \beta(x)\eta(x,1)\beta(x)^{-1}e\\ &= (x,\eta)e\end{aligned}$$

so φ and ψ are inverse homomorphisms. Therefore ψ and φ are isomorphisms.

Now suppose that G is perfect. We show that H can be chosen to be perfect. So we need $H = [H,H]$ where $[H,H]$ is the commutator subgroup of H. To do this, we let $L = [H,H]$ and note that $H = LZ(H)$, $Z(L) \subseteq Z(H) = Z$ and $[H,H] = [L,L]$. This gives us $L/Z(L) \cong$

$H/Z(H)$ so that L is still a central extension of G. Now for each $g \in G$ we can choose an $\eta_g \in Z$ such that $(g, \eta_g) \in L$. Let $f = \sum_{\zeta \in Z(L)} \zeta^{-1}(1, \zeta)$. Then f is a central idempotent of kL. Using $\varphi : kL \longrightarrow k_\lambda G$ defined by $\varphi((g, \eta)) = \beta(g)^{-1}\eta\hat{g}$ as above, we have $\varphi(f) = 1$ and φ is onto so that φ restricted to kLf is onto. Define $\psi : k_\lambda G \longrightarrow kLf$ by $\psi(\hat{g}) = \beta(g)\eta_g^{-1}(g, \eta_g)f$. Then as above we check that φ and ψ are inverse maps. So we may assume that $H = L$ is perfect. $\square$

Combining Theorem 3.2 and Theorem 3.4, $kGc \cong S \otimes_k kHe$ as k-algebras where H is a central extension of the group G/N. If we assume that N is a maximal normal subgroup, then we can assume that H is quasi-simple. If we further assume that P is a p-subgroup of G such that $P \cap N = \{1\}$, then we will again have an interior P-algebra homomorphism as in Proposition 3.3. We denote the image of P in G/N by $\overline{P}$ and note that $P \cong \overline{P}$.

Assume the setup of the above theorems. Recall that H is defined to be a central p'-extension of G/N by a finite cyclic p'-group $Z \leqslant k^\times$, where $H = ((G/N) \times Z)$ as sets and multiplication is given by $(xN, \eta)(yN, \zeta) = (syN, \eta\zeta\alpha(x, y))$ where α is a 2-cocycle with values in $k^\times$. Also recall that we defined e to be $\frac{1}{|Z|}\sum_{\eta \in Z} \eta^{-1}(1, \eta)$ and saw that e was a central idempotent of kH. Let τ be the map from kHe to $k_{\lambda^{-1}}(G/N)$ induced from $(xN, \eta)e \longmapsto \beta(x)^{-1}\eta\widehat{xN}$. For every $r \in P$ we denote by $\tilde{r}$ the unique element $(rN, \eta) \in H$ whose order is a power of p. If $\tilde{r} = (rN, \eta)$, then $\tau(\tilde{r}) = \beta(r)^{-1}\eta\widehat{rN}$. Since $\tilde{r}$ has prime power order, $\beta(r)^{-p^m}\eta^{p^m} = 1$ for some positive integer m and since k has characteristic p, $\beta(r)^{-1}\eta = 1$. Thus $\tau(\tilde{r}) = \widehat{rN}$.

Now let ψ be the map from $S \otimes_k k_{\lambda^{-1}}(G/N)$ to kGc defined by $s \otimes \widehat{gN} \longmapsto ss_g^{-1}g$. Then define $\phi : S \otimes_k kHe \longrightarrow kGc$ by $s \otimes he \longmapsto \psi(s \otimes \tau(he))$. We give $S \otimes_k kHe$ the structure of an interior P-algebra by defining the structural homomorphism to be the map $r \longmapsto s_r \otimes \tilde{r}e$. With this structural homomorphism it is easy to check that ϕ is an isomorphism of interior P-algebras.

4. More reduction results

Let G be a finite group and let b be a nilpotent block of kG with defect group P. Let N be a normal subgroup of G and suppose c is a defect zero block of kN such that c is G-stable and such that $cb \neq 0$. This implies that $cb = b$. In particular, we can write $c = b_1 + \cdots + b_t$ with $b_1 = b$ and the b_i all blocks of kG. If P is a defect group of b, then $P \cap N = 1$. Since c is defect zero, $S = kNc$ is isomorphic to a matrix algebra. So we can write $S = \mathrm{End}_k(U)$ for some k-vector space U. Let R be a maximal p-subgroup of G containing P such that $\mathrm{Br}_R(c) \neq 0$. Note that $\mathrm{Br}_R(c) \neq 0$ if and only if $\mathrm{Br}_R(b_i) \neq 0$ for some i. Also since c is a defect zero block, we have $R \cap N = 1$.

In Section 3, we showed that there is an interior R-algebra isomorphism $\psi : S \otimes_k kHe \longrightarrow kGc$. Where $S = kNc$ and H is a central p'-extension of G/N by a finite cyclic p'-group $Z \leqslant k^\times$. Recall that $H = ((G/N) \times Z)$ as sets and multiplication is given by extending $(xN, \eta)(yN, \zeta) = (xyN, \eta\zeta\alpha(x, y))$ where α is a 2-cocycle of G with values in Z. We also defined $e = \frac{1}{|Z|}\sum_{\zeta \in Z} \zeta^{-1}(1, \zeta)$ and saw that e is a central idempotent of kH.

By our choice of λ in Section 3, the map $r \longmapsto s_r$ from R to $S^\times$ is a group homomorphism. We also have a group homomorphism $r \longmapsto \tilde{r}$ from R to H. So for any subgroup Q of R we have isomorphic groups $\overline{Q}$ and $\widetilde{Q} = \{\tilde{r} \mid r \in Q \subset R\}$ in G/N and H respectively. Identify $\overline{Q}$ and $\widetilde{Q}$ with Q through these isomorphisms.

Define $\psi : S \otimes_k k_{\lambda^{-1}}(G/N) \longrightarrow kGc$ to be the homomorphism which sends $s \otimes \widehat{gN}$ to $ss_g^{-1}g$ for $s \in S$ and $g \in G$. Let τ be the homomorphism from kHe to $k_{\lambda^{-1}}(G/N)$ which extends $(sN, \eta) \longmapsto \beta(xN)^{-1}\eta\widehat{xN}$. Finally, define $\phi : S \otimes_k kHe \longrightarrow kGc$ by extending $\phi(s, he) =$

$\psi(s \otimes \tau(h))$. We showed that ϕ is a homomorphism of interior Q algebras for any $Q \leqslant R$ since $R \cap N = 1$. For any $Q \leqslant R$ we have an isomorphism of Brauer quotients $\phi_Q : S(Q) \otimes_k (kHe)(Q) \longrightarrow (kGc)(Q)$ such that $\phi_Q(\mathrm{Br}_Q(x) \otimes_k \mathrm{Br}_Q(y)) = \mathrm{Br}_Q(\phi(x \otimes y))$ by Proposition 28.3 of [15]. Also since S is simple we have a bijection between the blocks of kH that appear in e and those of kG which appear in c, given by $u \longmapsto \phi(1_S \otimes u)$. Let d be the block of kH which corresponds to b.

By the definition of R we have $S(Q) \neq 0$ for all $Q \leqslant R$. So $\mathrm{Br}_Q(b) \neq 0$ if and only if $\mathrm{Br}_{\widetilde{Q}}(d) \neq 0$. Theorem 5.16 in Chapter 5 of [11] implies that $\overline{R}$ is a Sylow p-subgroup of G/N. Therefore $\widetilde{R}$ is a Sylow p-subgroup of H. So $\widetilde{R}$ must contain a defect group of d, but the largest subgroup of $\widetilde{R}$ with non-zero image under the Brauer homomorphism is $\widetilde{P}$. Thus $\widetilde{P} \cong P$ is a defect group of d. This yields the following proposition.

Proposition 4.1. *With the above notation, the block d of kH that corresponds to b has a defect group isomorphic to P.*

Since S is simple, kGb and kHd are Morita equivalent. Therefore there is a correspondence between the simple modules of kHd and those of $kGb \cong S \otimes_k kHd$ given by $V \longmapsto U \otimes V$. Also since b is nilpotent, kHb has a unique isomorphism class of simple modules. So if M is the unique simple kGb module and V is the unique simple kHd-module, then $M \cong U \otimes V$. A vertex of V is contained in $\widetilde{P}$ since $\widetilde{P}$ is a defect group of d. Suppose V has vertex $\widetilde{Q} \subseteq \widetilde{P}$ with source X. By our choice of R, $S(Q) \neq 0$. So we know that U must contain a summand of vertex Q. Therefore $U \otimes X$ has a summand of vertex Q, which implies that $U \otimes X$ is a direct summand of $\mathrm{Res}_Q(U \otimes V) \cong \mathrm{Res}_Q(M)$ and $U \otimes X$ has a summand of vertex Q. So Q contains a vertex of M. Therefore $\widetilde{Q} = \widetilde{P}$ and the source of M is isomorphic to a direct summand of the kP-module $U \otimes X$.

Finally we show that d is nilpotent. This will allow us to reduce our problem from G to H. The following can be found in [9]. If Q is a subgroup of R, then $S(Q) \neq 0$. So $S(Q)$ is a matrix algebra by 28.6 of [15]. This means that ϕ_Q is an isomorphism from $S(Q) \otimes_k (kHe)(Q)$ to $(kGc)(Q)$. Also notice that $(kHe)(Q) \cong kC_H(Q)\,\mathrm{Br}_Q(e)$ and $(kGc)(Q) \cong kC_G(Q)\,\mathrm{Br}_Q(c)$. Therefore $f \longmapsto \phi_Q(1 \otimes f)$ gives a bijection between the set of blocks of $kC_H(Q)\,\mathrm{Br}_Q(e)$ and the set of blocks of $kC_G(Q)\,\mathrm{Br}_Q(c)$. If we consider this map over all $Q \leqslant P$, then we get a bijection between the set of Brauer pairs of blocks of kH appearing in e and the set of blocks of kG appearing in c. In particular, we have a bijection between the d-Brauer pairs and the b-Brauer pairs. The following theorem shows that d is nilpotent and a proof can be found in 3.5 of [9].

Theorem 4.2. *Let (Q, v) be a d-Brauer pair with $Q \leqslant P$ and let $u = \phi(1 \otimes v)$ so that (Q, u) is a b-Brauer pair. Then $N_H(Q, v)$ and $N_G(Q, u)$ have the same image in* $\mathrm{Aut}(Q)$. *In particular d is nilpotent.*

5. Main reduction

We start by proving the following.

Theorem 5.1. *Let P be a finite p-group. Suppose that there is a finite group H with P a subgroup of* $\mathrm{Aut}(H)$. *Also suppose that c is a defect zero block of kH such that c is P-stable and* $\mathrm{Br}_P(c) \neq 0$. *Finally, suppose that, when viewed as a block of $k(H \rtimes P)$, c has a non-torsion*

kP-module as a source of its unique (up to isomorphism) simple module. Then H can be chosen to be a central p'-extension of a direct product of isomorphic simple groups.

The proof requires the following.

Lemma 5.2. *Assume the notation of Theorem* 5.1. *Let N be a normal subgroup of H which is maximal with respect to being P-stable. Then we can find a P-stable block f of kN such that $cf \neq 0$.*

Proof. It is clear that there exists a block d of kN such that $cd \neq 0$. Let $G = \mathrm{Stab}_{(H \rtimes P)}(d) = \{x \in H \rtimes P \mid xdx^{-1} = d\}$ and let $e = \mathrm{Tr}_G^{H \rtimes P}(d)$. Then e is the sum of distinct blocks of kN. So e is a central idempotent of $k(H \rtimes P)$. Therefore e is a sum of blocks of $k(H \rtimes P)$ and one of these blocks is c. So $ce = c$ which implies that $\mathrm{Br}_P(e) \neq 0$, since $\mathrm{Br}_P(e)\,\mathrm{Br}_P(c) = \mathrm{Br}_P(c) \neq 0$. Mackey's theorem tells us that $e = \mathrm{Tr}_G^{H \rtimes P}(d) = \sum_{y \in [P \backslash (H \rtimes P)/G]} \mathrm{Tr}_{P \cap {}^yG}^P({}^yd)$. If $P \cap {}^yG \subsetneq P$ for all $y \in [P\backslash(H \rtimes P)/G]$, then $\mathrm{Br}_P(e) = 0$. So we must have a y such that $P \cap {}^yG = P$. By letting $f = {}^yd$ we are done since $\mathrm{Stab}_{(H \rtimes P)}(f) = {}^y\mathrm{Stab}_{H \rtimes P}(d)$. $\square$

Proof of Theorem 5.1. We assume the setup of Theorem 5.1 and let N be a normal subgroup of H which is maximal with respect to being P stable. Let d be a block of kN such that $cd \neq 0$. The lemma allows us to assume that $P \subseteq \mathrm{Stab}_{(H \rtimes P)}(d)$. This implies that $\mathrm{Stab}_{(H \rtimes P)}(d) = \mathrm{Stab}_H(d)P \cong \mathrm{Stab}_H(d) \rtimes P$. Now we can apply Proposition 3.1 to c as a block of $k(H \rtimes P)$. We get a block e of $k\,\mathrm{Stab}_{(H \rtimes P)}(d)$ that satisfies all the conditions of the proposition and the source of the simple module of $k\,\mathrm{Stab}_{(H \rtimes P)}(d)e$ is the same as that of $k(H \rtimes P)c$. View e as a block of $k(\mathrm{Stab}_H(d) \rtimes P)$. Then it is easy to see that e is a block of $k(\mathrm{Stab}_H(d))$. So we may assume that $H = \mathrm{Stab}_H(d)$. Thus we can assume that d is stable in $H \rtimes P$.

Having d be $H \rtimes P$ stable allows us to apply our main reduction Theorem 3.2 and its generalizations at the end of Section 3. So we can find a central extension G of $(H \rtimes P)/N$ with a cyclic p' center Z and a block e of kG with the following properties. The block e is nilpotent with defect P and the source of the simple kGe-module is a source of the simple $k(H \rtimes P)c$-module. Finally we need to show that G is isomorphic to a semi-direct product of H/N with P.

Let G be as above and let λ be the 2-cocycle described in Section 3. We let $\widetilde{G}$ be the group whose underlying set is $(H/N) \times Z$ and with multiplication defined by $(hN, \eta)(gN, \zeta) = (hgN, \lambda(h, g)\eta\zeta)$. We also define an action of P on $\widetilde{G}$ by ${}^u(hN, \eta) = (uhu^{-1}N, \lambda(u, h)\eta)$ for $u \in P$. Now let $\varphi : G \longrightarrow \widetilde{G} \rtimes P$ be defined by extending $(hpN, \eta) \longmapsto (hN, \eta)p$. This is clearly a bijection. So we just need to show φ is a homomorphism. To do this observe that for $h, g \in H$, $p, q \in P$ and $\eta, \zeta \in Z$

$$\begin{aligned}
\varphi\big((hpN, \eta)(gqN, \zeta)\big) &= \varphi\big(\big(hpgp^{-1}pqN, \lambda(hp, gq)\eta\zeta\big)\big) \\
&= \big(hpgp^{-1}N, \lambda(hp, gq)\eta\zeta\big)pq
\end{aligned}$$

and that

$$\begin{aligned}
\varphi\big((hpN, \eta)\big)\varphi\big((gqN, \zeta)\big) &= (hN, \eta)p(gN, \zeta)q \\
&= (hN, \zeta)\big(pgp^{-1}N, \lambda(p, g)\zeta\big)pq \\
&= \big(hpgp^{-1}N, \lambda\big(h, pgp^{-1}\big)\big)\lambda(p, g\eta\zeta)pq.
\end{aligned}$$

So to prove Theorem 5.1 we need $\lambda(hp, gq) = \lambda(h, pgp^{-1})\lambda(p, g)$. Recall that by our choice of λ we have $\lambda(g, p) = 1$ for all $g \in H$ and all $p \in P$. The rest follows from the structural equations for 2-cocycles. So we can replace G by $\widetilde{G} \rtimes P$.

Finally, since N was maximal with respect to being normal in H and P-stable, we have that H/N is a minimal normal subgroup of $(H/N) \rtimes P$. Therefore H/N is a direct product of isomorphic simple groups. $\square$

If H/N is a direct product of cyclic groups, then the central extension of H/N is p-solvable. A result of Puig shows that the source of a simple module of a block of a p-solvable group is torsion in $D(P)$. So we may assume that H/N is a direct product of non-abelian simple groups. In this case the simple groups will be transitively permuted by P.

Now we eliminate the direct product from Theorem 5.1. Assume that H is a central p'-extension of $S_1 \times \cdots \times S_t$ by Z with all of the S_i isomorphic to the non-abelian simple group S_1. We also suppose that c is a defect zero block of kH which is P-stable for a p-group $P \subseteq \mathrm{Aut}(H)$. Finally assume that a simple kHc-module has a non-torsion source when viewed as a $k(H \rtimes P)c$-module. Let φ be the canonical surjection of H onto $S_1 \times \cdots \times S_t$ and let $D_i = \varphi^{-1}(S_i)$. Then each D_i is isomorphic to a central p'-extension of S_i by Z. It is also easy to see that for $x \in D_i$ and $y \in D_j$ with $i \neq j$ we have $[x, y] = xyx^{-1}y^{-1} \in Z$. Fix an $x \in D_i$ for some i. Then the map from D_j to Z defined by $y \longmapsto [x, y]$ for $j \neq i$ is a group homomorphism. It is also easy to see that this map factors through S_j. Since S_j is simple, the map must be 1. Therefore $[D_i, D_j] = 1$ for all $i \neq j$. So if $x \in D_i$ and $y \in D_j$ with $i \neq j$, we have $xy = yx$.

Define a surjective map $\varphi : D_1 \times \cdots \times D_t \longrightarrow H$ by $(d_1, \dots, d_t) \longmapsto d_1 \dots d_t$ for $d_i \in D_i$. Since elements of D_i and D_j commute for $i \neq j$, φ is a homomorphism. We now pull everything back to $D_1 \times \cdots \times D_t$. By a general lifting theorem on idempotents, Theorem 3.2 in [15], we can find a block f of $k(D_1 \times \cdots \times D_t)$ whose image in kH is c. It is easy to see that $\ker(\varphi)$ is a p'-group. So we can apply Theorem 8.8 of Chapter 5 in [11] and see that $k(D_1 \times \cdots \times D_t)f \cong kHc$. Moreover, if M is a simple kHc-module, then M will be a simple $k(D_1 \times \cdots \times D_t)f$-module with the action via the above homomorphism.

Note that $k(D_1 \times \cdots \times D_t) \cong (kD_1) \otimes_k \cdots \otimes_k (kD_t)$. So there are unique defect zero blocks f_i of kD_i such that $(kD_1 f_1) \otimes_k \cdots \otimes_k (kD_t f_t) \cong k(D_1 \times \cdots \times D_t)f \cong kHc$. We can define a P algebra structure on $(kD_1 f_1) \otimes_k \cdots \otimes_k (kD_t f_t)$ which agrees with the action of P on kHc. Also notice that, since Z commutes with the action of P and the S_i are transitively permuted by P, the D_i are transitively permuted by P.

Set $Q = \mathrm{Stab}_P(D_1) = \mathrm{Stab}_P(S_1)$. Then $t = [P : Q]$. We choose $\{u_1, \dots, u_t\} = [P/Q]$ with $u_1 = 1_Q$. Also assume that the action of u_i maps D_1 to D_i. From this we find a P-algebra structure on $(kD_1 f_1) \otimes_k \cdots \otimes_k (kD_t f_t)$ such that the action of P permutes the $1 \otimes \cdots \otimes (kD_i f_i) \otimes \cdots \otimes 1$. We show that as P-algebras $(kD_1 f_1) \otimes_k \cdots \otimes_k (kD_t f_t)$ is isomorphic to $\mathrm{Ten}_Q^P(kD_1 f_1)$.

Let Q be a subgroup of a p-group P and let B be a Q-algebra. Recall $\mathrm{Ten}_Q^P(B) = B \otimes_k \cdots \otimes_k B$ as k-algebras, where the number of B's above is $[P : Q]$. The action of P on $\mathrm{Ten}_Q^P(B)$ is defined as follows. Let $b_1, \dots, b_t \in B$ and $v \in P$. Then ${}^{v}(b_1, \dots, b_t) = ({}^{q_{\tau^{-1}(1)}}b_{\tau^{-1}(1)}, \dots, {}^{q_{\tau^{-1}(t)}}b_{\tau^{-1}(t)})$ where $\tau \in S_t$ and $q_i \in Q$ are defined by $vu_i = u_{\tau(i)}q_i$. Now let $A = A_1 \otimes_k \cdots \otimes_k A_t$ be a P-algebra such that P transitively permutes the $1 \otimes \cdots \otimes A_i \otimes \cdots \otimes 1$ and identify $1 \otimes \cdots \otimes A_i \otimes \cdots 1$ with A_i. Let $Q = \mathrm{Stab}_P(A_1)$ and let $\{u_1, \dots, u_t\} = [P/Q]$ be a set of coset representatives such that the action of u_i maps A_1 to A_i. Next identify A_i with A_1

via the isomorphism induced by u_i. In this way we can assume that ${}^{u_i}(a \otimes 1 \otimes \cdots \otimes 1) = (1 \otimes \cdots \otimes a \otimes \cdots \otimes 1)$ where a is in the ith position.

Let $v \in P$ and let $1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ be an arbitrary element of A_j for some j. We can write $vu_j = u_i w$ for unique i and $w \in Q$. Then the action of v on $1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ is the same as the action of vu_j on $a \otimes 1 \otimes \cdots \otimes 1$. This is the same as the action of $u_i w$ which sends $a \otimes 1 \otimes \cdots \otimes 1$ to $1 \otimes \cdots \otimes {}^w a \otimes \cdots \otimes 1$ where ${}^w a$ is in the ith position. This implies that $A \cong \mathrm{Ten}_Q^P(A_1)$.

Letting $B = kD_1 f_1$ and $A = kHc$, the above yields $kHc \cong \mathrm{Ten}_Q^P(kD_1 f_1)$ as P-algebras. We know that $kD_1 f_1$ is a Q-algebra and it is isomorphic to $\mathrm{End}_k(M)$ where M is a simple $kD_1 f_1$-module. Then by Lemma 2.1 of [2] we know that $\mathrm{Ten}_Q^P(\mathrm{End}_k(M))$ is isomorphic to $\mathrm{End}_k(\mathrm{Ten}_Q^P(M))$ as P-algebras. Thus we have that a simple kHc-module is isomorphic to $\mathrm{Ten}_Q^P(M)$. Note that if $P = Q$, then H is a central p'-extension of a simple group.

Now we assume that $P \cong C_p \times C_p$. Suppose that $Q \subsetneq P$ and let V be a kQ-module which is a source of M as a $k(D_1 \rtimes Q) f_1$-module. Then V is torsion in $D(Q)$ since Q is trivial or cyclic and $D(Q)$ is finite in these cases. Since V is a source of M, $M \cong V \oplus X$ as kQ-modules for some kQ-module X, and we know that V has vertex Q. Then by Proposition 3.15.2 of [1], $\mathrm{Ten}_Q^P(M) \cong \mathrm{Ten}_Q^P(V \oplus X) \cong \mathrm{Ten}_Q^P(V) \oplus \mathrm{Ten}_Q^P(X) \oplus U$ as kP-modules, where U is the direct sum of modules induced from proper subgroups of Q. We also know that $\mathrm{Ten}_Q^P(V)$ has a summand of vertex P. So if W is a source of a simple kHc-module, then $[W] = [\mathrm{Ten}_Q^P(V)]$ in $D(P)$. Also since $\mathrm{Ten}_Q^P : D(Q) \longrightarrow D(P)$ is a group homomorphism we conclude that W must also be torsion in $D(P)$, since V is torsion in $D(Q)$. Therefore we can assume that H is a central p'-extension of a finite simple group. Thus we have established the following reduction.

Theorem 5.3. *Let $P \cong C_p \times C_p$ where p is a prime. Suppose there is a finite group H with P a subgroup of* $\mathrm{Aut}(H)$. *Also suppose that c is a defect zero block of kH such that c is P-stable and* $\mathrm{Br}_P(c) \neq 0$. *Finally, suppose that, when viewed as a block of $k(H \rtimes P)$, c has a non-torsion kP-module as a source of its unique (up to isomorphism) simple module. Then H can be chosen to be a central p'-extension of a simple group.*

6. Simple groups

Suppose that H is a finite group and P is a subgroup of $\mathrm{Aut}(H)$. Also assume that c is a P-stable defect zero block of kH such that $\mathrm{Br}_P(c) \neq 0$ and that $u \in P \cap \mathrm{Inn}(H)$ is nontrivial. Then there exists an $x \in H$ such that ${}^u h = {}^x h$ for all $h \in H$. Let $c = \sum_{h \in H} \lambda_h h$. Since $\mathrm{Br}_P(c) \neq 0$, there exists an $h_0 \in H$ such that ${}^u h_0 = h_0$ and $\lambda_{h_0} \neq 0$. Therefore ${}^x h_0$ is also equal to h_0. Since $|u| = p^e$ for some positive integer e, we have $x^{p^e} \in Z(H)$ and $|x| = p^e m$ for some positive integer m. Then $|x^m| = p^e$ and x^m stabilizes h_0. So $\langle x^m \rangle$ is a p-group and $\mathrm{Br}_{\langle x^m \rangle}(c) \neq 0$. This implies that $\langle x^m \rangle$ is contained in a defect group of c, but c is a defect zero block. Therefore no such u exists and this yields following.

Proposition 6.1. *Let H be a finite group, let P be a p-group in* $\mathrm{Aut}(H)$ *and let c be a defect zero block of kH which is P-stable and such that* $\mathrm{Br}_P(c) \neq 0$. *Then* $P \cap \mathrm{Inn}(H) = 1$. *In particular, P will be isomorphic to a subgroup of* $\mathrm{Out}(H)$.

So we can detect P inside of $\mathrm{Out}(H)$. From Section 5 we may assume that H is a central p'-extension of a simple group. Also if H is a central p'-extension of the finite group G it is known that $\mathrm{Out}(H)$ is a subgroup of $\mathrm{Out}(G)$. So we consider $\mathrm{Out}(G)$ for the finite simple groups G. We are concerned with when $C_p \times C_p$ will be a subgroup of $\mathrm{Out}(G)$.

We now run through the classification of the finite simple groups and check to see when $C_p \times C_p$ is a subgroup of $\mathrm{Out}(G)$ for a simple group G. We will see that when p is odd, the set of groups where this can happen is very small. First consider the alternating groups. It is well known that for $n \geqslant 5$ and $n \neq 6$, $\mathrm{Aut}(A_n) \cong S_n$. So $|\mathrm{Out}(A_n)| = 2$. We also note that $A_6 \cong PSL_2(9)$ and we will deal with this case later. Therefore $C_p \times C_p$ is not a subgroup of $\mathrm{Out}(A_n)$ for any $n \geqslant 5$ and $n \neq 6$. We can also eliminate the sporadic groups from the list of groups we need to check since $|\mathrm{Out}(G)| \leqslant 2$ for all 26 sporadic groups. So if $C_p \times C_p$ is a subgroup of $\mathrm{Out}(G)$ for a simple group G, then G must be a finite group of Lie type.

We now look at the finite simple groups of Lie type. A more complete treatment of the following can be found in [6]. The Chevalley groups, also known as the untwisted groups of Lie type, are denoted by $A_n(q)$, $B_n(q)$ $(n \geqslant 2)$, $C_n(q)$ $(n \geqslant 3)$, $D_n(q)$ $(n \geqslant 4)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$ and $G_2(q)$, where $q = r^f$ for some prime r and positive integers f and n satisfying the given condition, if any. The twisted Chevalley groups, also called the Steinberg groups, are denoted by ${}^2A_n(q)$ $(n \geqslant 2)$, ${}^2D_n(q)$ $(n \geqslant 3)$, ${}^3D_4(q)$ and ${}^2E_6(q)$ with q and n as above. Finally the Suzuki–Ree groups are denoted by ${}^2B_2(2^{2f+1})$, ${}^2F_4(2^{2f+1})$ and ${}^2G_2(3^{2f+1})$. With this notation we have the following.

Theorem 6.2. *(See [6].) Let G be the adjoint version of a finite group of Lie type. Then G is a non-abelian simple group except when*

$$G = A_1(2), A_1(3), {}^2A_2(2), {}^2B_2(2), B_2(2), G_2(2), {}^2F_4(2), \text{ and } {}^2G_2(3)$$

and for $G = B_2(2), G_2(2), {}^2F_4(2)$ and ${}^2G_2(3)$ the commutator $[G, G]$ is the unique non-trivial proper normal subgroup. Moreover, in these cases $[G, G]$ is a non-abelian simple group.

Three of the simple groups which show up as commutators in the above theorem are isomorphic to other simple groups in the list of groups of Lie type via the following isomorphisms $A_1(8) \cong ({}^2G_2(3))'$, $A_1(9) \cong B_2(2)'$, and ${}^2A_2(3) \cong G_2(2)'$. However, there is no such isomorphism for $({}^2F_4(2))'$ and this is a simple group that does not show up elsewhere in the classification. This leads to the following definition.

Definition 6.3. (See [6].) The *finite simple groups of Lie type* are $({}^2F_4(2))'$ and the finite groups of Lie type excluding

$$G = A_1(2), A_1(3), {}^2A_2(2), {}^2B_2(2), B_2(2), {}^2F_4(2) \text{ and } {}^2G_2(3).$$

We first note that $\mathrm{Out}({}^2F_4(2)') = C_2$. Thus P is not a subgroup of $\mathrm{Out}({}^2F_4(2)')$. So we will only need to look at the finite simple groups of Lie type which are actually finite groups of Lie type.

If G is an adjoint version of a finite group of Lie type, then $\mathrm{Out}(G)$ will have order dfg where d, f and g are the orders of certain subgroups of $\mathrm{Out}(G)$. d is the order of the subgroup of diagonal automorphisms. This group is cyclic or elementary abelian of order 4. The field automorphisms will form a subgroup of order f and this subgroup will always be cyclic. Finally, the subgroup of order g is the group of graph automorphisms which is cyclic of order 2 or trivial

Table 1
Orders of Out(G) for the adjoint versions for the groups of Lie type [3]

Conditions	G	d	f	g
$n \geqslant 2$	$A_1(q)$	$(2, q-1)$	$q = r^f$	1
$n \geqslant 2$	$A_n(q)$	$(n+1, q-1)$	$q = r^f$	2
	${}^2A_n(q)$	$(n+1, q+1)$	$q^2 = r^f$	1
f odd	$B_2(q)$	$(2, q-1)$	$q = r^f$	2 if $r = 2$
$n \geqslant 3$	${}^2B_2(q)$	1	$q = 2^f$	1
	$B_n(q)$	$(2, q-1)$	$q = r^f$	1
$n \geqslant 3$	$C_n(q)$	$(2, q-1)$	$q = r^f$	1
$n > 4$, even	$D_4(q)$	$(2, q-1)^2$	$q = r^f$	3!
$n > 4$, odd	${}^3D_4(q)$	1	$q^3 = r^f$	1
$n \geqslant 4$	$D_n(q)$	$(2, q-1)^2$	$q = r^f$	2
	$D_n(q)$	$(4, q^n-1)$	$q = r^f$	2
	${}^2D_n(q)$	$(4, q^n+1)$	$q^2 = r^f$	1
f odd	$G_2(q)$	1	$q = r^f$	2 if $r = 3$
	${}^2G_2(q)$	1	$q = 3^f$	1
f odd	$F_4(q)$	1	$q = r^f$	2 if $r = 2$
	${}^2F_2(q)$	1	$q = 2^f$	1
	$E_6(q)$	$(3, q-1)$	$q = r^f$	2
	${}^2E_6(q)$	$(3, q+1)$	$q^2 = r^f$	1
	$E_7(q)$	$(2, q-1)$	$q = r^f$	1
	$E_8(q)$	1	$q = r^f$	1

(unless $G = D_4(q)$ in which case it is isomorphic to S_3). In Table 1, we list d, f and g for the groups of Lie type. A similar table is given in [3].

Next we look at the question of when is a non-cyclic p-group P a subgroup of Out(G). Since P is not cyclic and p is odd we must have p dividing at least two of the d, f and g. Let $q = r^f$ where r is the prime over which the group of Lie type is defined. We now restrict to the case of p being odd. When $p \geqslant 5$ there are only two types of groups where this happens. First we could have $A_n(q)$ with $p \mid (n+1, q-1)$ (notice that this means $p \neq r$) and $p \mid f$. The second case that we cannot eliminate is ${}^2A_n(q)$ with $p \mid (n+1, q+1)$ (again this means $p \neq r$) and $p \mid f$. When $p = 3$ we can again eliminate most cases. In this case there are only five types of groups that can show up. As above we can have $A_n(q)$ with $p \mid (n+1, q-1)$ and $p \mid f$ or ${}^2A_n(q)$ with $p \mid (n+1, q+1)$ and $p \mid f$. We also have the cases $E_6(q)$ with $p \mid (q-1)$ and $p \mid f$ and ${}^2E_6(q)$ with $p \mid (q+1)$ and $p \mid f$. In all four of these cases we must have $p \neq r$. Our final possibility is $D_4(q)$ with $p \mid f$ and in this case we cannot assume $p \neq r$. In the next section we eliminate the cases of $G = E_6(q)$ and $G = {}^2E_6(q)$ by using Aut(G) instead of just Out(G).

7. $E_6(q^3)$ and ${}^2E_6(q^3)$

Let G be the adjoint versions of $E_6(q^3)$ or ${}^2E_6(q^3)$ and let $\widehat{G}$ denote the universal versions. In this section, we show that if $3 \mid (q-1)$ for $E_6(q^3)$ or $3 \mid (q+1)$ for ${}^2E_6(q^3)$, then $P \cong C_3 \times C_3$ is never a subgroup of Aut(G) such that $P \cap \mathrm{Inn}(G) = 1$. Our first step is to find an element h

of $\widehat{G}$ such that h is not fixed by the standard field automorphism σ of order 3 but the image of h in G is fixed by σ. In order to do this we need to introduce the Cartan subgroup $\widehat{H}$ of $\widehat{G}$. This can be done for all finite groups of Lie type, but we will just introduce the facts that we need. First we will state some known results. It is known that $Z(\widehat{G}) \subseteq \widehat{H}$ (see Proposition 2.5.9 of [6]). Let $\{\alpha_1, \ldots, \alpha_s\}$ be a set of fundamental roots. So $s = 6$ if $G = E_6(q^3)$ and $s = 4$ if $G = {}^2E_6(q^3)$. Then there are subgroups $\widehat{H_i} = \langle h_{\alpha_i}(t)\rangle$ of $\widehat{H}$ such that the following hold. When $G = E_6(q^3)$, $\widehat{H_i} \cong (GF(q^3))^\times \cong C_{(q^3-1)}$ for $i \in \{1, \ldots, 6\}$ and when $G = {}^2E_6(q^3)$, $\widehat{H_1} \cong \widehat{H_2} \cong (GF(q^{3*2}))^\times \cong C_{(q^{3*2}-1)}$ and $\widehat{H_3} \cong \widehat{H_4} \cong (GF(q^3))^\times \cong C_{(q^3-1)}$. Also by Theorem 2.4.7 of [6], $\widehat{H} \cong \{(h_1, \ldots, h_s)\colon\ h_i \in \widehat{H_i}\}$. One of the main results we need describes how the standard field automorphism of order 3 will act on $\widehat{H}$.

Theorem 7.1. *(See [6, 2.5.1].) Let σ be the standard field automorphism of order 3. We identify each $\widehat{H_i}$ with the appropriate Galois field. Then σ acts on $\widehat{H}$ as $\sigma(h_1, \ldots, h_s) = (h_1^q, \ldots, h_s^q)$.*

We also need the following lemma.

Lemma 7.2. *Let $Z(\widehat{G}) = \langle z\rangle$. There exists $w \in \widehat{H}$ such that $w^{-1}\sigma(w) = z$.*

Proof. Proposition 2.5.9 of [6] tells us that $z = (z_1, \ldots, z_s) \in \widehat{H}$. We also know that z has order 3. If $3 \mid (q+1)$, then 3 does not divide $q^3 - 1$. So in this case $z_3 = z_4 = 1$. Note that $3 \mid (q-1)$ implies $3(q-1) \mid q^3 - 1$ and $3 \mid q+1$ implies $3(q-1) \mid q^6 - 1$. This means that for each $z_i \in H_i$ we can find a $w_i \in H_i$ such that $w_i^{(q-1)} = z_i$. Letting $w = (w_1, \ldots, w_s)$ completes the proof of the lemma since $w^{-1}\sigma(w) = w^{-1}(w_1^q, \ldots, w_s^q) = (w_1^{(q-1)}, \ldots, w_s^{(q-1)}) = z$. $\square$

The important fact about w is that w will not be fixed by σ in $\widehat{G}$, but the image of w in G will be fixed by σ. Now that we have found w we will use some known facts on the centralizers of σ in $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\widehat{G})$ to obtain our result. In an earlier section we said that $\mathrm{Out}(G)$ will have a cyclic subgroup of order 3 which we called the diagonal subgroup. The pre-image of the diagonal subgroup is the group of *inner diagonal* automorphisms of G and is denoted by $\mathrm{Inndiag}(G)$. So $\mathrm{Inndiag}(G) \cong G\langle\gamma\rangle$ where $\gamma \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$. We can now state the following.

Theorem 7.3. *(See [6, 4.9.1].) Let $K = E_6(q^3)$ or ${}^2E_6(q^3)$, let σ be a field automorphism of K of order 3 and let $K_\sigma = E_6(q)$ or ${}^2E_6(q)$ respectively. Here we do not restrict the versions of K. Then the following hold*

(i) *If $y \in \mathrm{Inndiag}(K)\sigma$ has order 3, then σ and y are $\mathrm{Inndiag}(K)$-conjugate,*
(ii) *If K is adjoint, then so is K_σ and $C_{\mathrm{Inndiag}(K)}(\sigma) \cong \mathrm{Inndiag}(K_\sigma)$,*
(iii) *If K is universal, then $C_K(\sigma) \cong K_\sigma$ and K_σ is universal.*

Now we can state the main result of this section.

Theorem 7.4. *Let $G = E_6(q^3)$ with $3 \mid (q-1)$ or ${}^2E_6(q^3)$ with $3 \mid (q+1)$. We view $G \cong \mathrm{Inn}(G)$ as a subgroup of $A = \mathrm{Aut}(G)$. Then there is no subgroup P of A with $P \cong C_3 \times C_3$ and $P \cap G = 1$.*

Proof. Suppose that P is a subgroup of A such that $P \cong C_p \times C_p$ and $P \cap G = 1$. Since $\mathrm{Out}(G)$ has 3-rank 2, $P \cap G\sigma$ must be non-empty. Now by (i) of Theorem 7.3 there is a $g \in \mathrm{Inndiag}(G)$ such that $\sigma \in {}^gP$ and ${}^gP \cap G = 1$. So by replacing P with ${}^gP$ we may assume that $\sigma \in P$.

Now applying (iii) of Theorem 7.3 $C_{\widehat{G}}(\sigma) \cong \widehat{G}_\sigma$ and $\widehat{G}_\sigma$ is isomorphic to the universal version of $E_6(q)$ or ${}^2E_(q)$. By our comments on Inndiag we have that $\mathrm{Inndiag}(G_\sigma) = G_\sigma\langle\gamma\rangle$ for some $\gamma \in \mathrm{Aut}(G_\sigma) \setminus \mathrm{Inn}(G_\sigma)$. Then by Lemma 7.2 we found an element $w \in \widehat{G}$ which is not fixed by σ but the image of w, denoted $\bar{w}$, in G is fixed by σ. So by passing from $\widehat{G}$ to G we find a $\bar{w} \in G$ such that $\bar{w} \in C_G(\sigma)$ and $\bar{w} \notin G_\sigma$. Now by (ii) of Theorem 7.3 we have $C_{\mathrm{Inndiag}(G)}(\sigma) \cong \mathrm{Inndiag}(G_\sigma)$ and since $\mathrm{Inndiag}(G_\sigma) = G_\sigma(\gamma)$ for some γ and we know that $\bar{w} \in G \setminus G_\sigma$. So $\mathrm{Inndiag}(G_\sigma) \cong G_\sigma\langle\bar{w}\rangle$. Therefore $C_{\mathrm{Inndiag}(G)}(\sigma) \subseteq G$. Therefore $P \cap G$ is not empty. So no such P exists and the result is shown. □

Combining this result with Section 6 we have the following corollary to Theorem 5.3.

Corollary 7.5. *Assume that p is odd. Suppose that $P = C_p \times C_p \leqslant \mathrm{Aut}(G)$ where G is a finite group. Suppose that b is a P-stable defect 0 block of kG such that $\mathrm{Br}_P(b) \neq 0$. Finally, suppose that the source V of a simple $k(G \rtimes P)b$-module M is a finitely generated endo-permutation kP-module whose image is non-torsion in $D(P)$. Then we can find G, b, V where G is one of the following.*

(i) (a) *G is a central p'-extension of $A_n(q)$ with $p \mid (n+1, q-1, f)$ where $q = r^f$; or*
(b) *G is a central p'-extension of ${}^2A_n(q)$ with $p \mid (n+1, q+1, f)$ where $q = r^f$; or*
(ii) *$p = 3$, $q = r^f$ and G is a central extension of $D_4(q)$ with $3 \mid f$.*

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